Basic Mathematics

The Product and Quotient Rules

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The aim of this package is to provide a short self assessment programme for students who want to learn how to use the product and quotient rules of differentiation.

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1. Basic Results

Differentiation is a very powerful mathematical tool. This package reviews two rules which let us calculate the derivatives of products of functions and also of ratios of functions. The rules are given without any proof.

It is convenient to list here the derivatives of some simple functions:

y	ax^n	$\sin(ax)$	$\cos(ax)$	e^{ax}	$\ln(x)$
$\dfrac{dy}{dx}$	nax^{n-1}	$a\cos(ax)$	$-a\sin(ax)$	$a\mathrm{e}^{ax}$	$\dfrac{1}{x}$

Also recall the **Sum Rule:**

$$\frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx}$$

This simply states that the derivative of the sum of two (or more) functions is given by the sum of their derivatives.

It should also be recalled that derivatives commute with constants:

$$\text{i.e., if} \quad y = af(x)\,, \quad \text{then} \quad \frac{dy}{dx} = a\frac{df}{dx}$$

where a is any constant.

Exercise 1. Differentiate the following with respect to x using the above rules (click on the **green** letters for the solutions).

(a) $y = 4x^2 + 3x - 5$ (b) $y = 4\sin(3x)$

(c) $y = \mathrm{e}^{-2x}$ (d) $y = \ln\left(\dfrac{x}{2}\right)$

Quiz Select the derivative of $y = \frac{1}{3}\mathrm{e}^{3t} - 3\cos\left(\dfrac{2t}{3}\right)$ with respect to t.

(a) $3\mathrm{e}^{3t} - 2\cos\left(\dfrac{2t}{3}\right)$ (b) $e^{3t} + 2\sin(t)$

(c) $\mathrm{e}^{3t} + 2\sin\left(\dfrac{2t}{3}\right)$ (d) $\mathrm{e}^{3t} - 2\sin\left(\dfrac{2t}{3}\right)$

2. The Product Rule

The **product rule** states that if u and v are both functions of x and y is their product, then the derivative of y is given by

$$\boxed{\text{if} \quad y = uv\,, \quad \text{then} \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}$$

Here is a **systematic procedure** for applying the product rule:

- Factorise y into $y = uv$;
- Calculate the derivatives $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$;
- Insert these results into the product rule;
- Finally perform any possible simplifications.

Example 1 The product rule can be used to calculate the derivative of $y = x^2 \sin(x)$. First recognise that y may be written as $y = uv$, where u, v and their derivatives are given by :

$$u = x^2 \qquad\qquad v = \sin(x)$$

$$\frac{du}{dx} = 2x \qquad\qquad \frac{dv}{dx} = \cos(x)$$

Inserting this into the **product rule** yields:

$$\begin{aligned} \frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= x^2 \times \cos(x) + \sin(x) \times (2x) \\ &= x^2 \cos(x) + 2x \sin(x) \\ &= x(x \cos(x) + 2 \sin(x)) \end{aligned}$$

where the common factor of x has been extracted.

Exercise 2. Use the product rule to differentiate the following products of functions with respect to x (click on the **green** letters for the solutions).

(a) $y = uv\,,$ if $u = x^m\,,$ and $v = x^n$

(b) $y = uv\,,$ if $u = 3x^4\,,$ and $v = \mathrm{e}^{-2x}$

(c) $y = uv\,,$ if $u = x^3\,,$ and $v = \cos(x)$

(d) $y = uv\,,$ if $u = \mathrm{e}^x\,,$ and $v = \ln(x)$

Exercise 3. Use the product rule to differentiate the following with respect to x (click on the **green** letters for the solutions).

(a) $y = x\mathrm{e}^{2x}$

(b) $y = \sin(x)\cos(2x)$

(c) $y = x\ln(4x^2)$

(d) $y = \sqrt{x}\,\ln(x)$

3. The Quotient Rule

The **quotient rule** states that if u and v are both functions of x and y then

$$\boxed{\text{if} \quad y = \frac{u}{v}\,, \quad \text{then} \quad \frac{dy}{dx} = \frac{\left(v\dfrac{du}{dx} - u\dfrac{dv}{dx}\right)}{v^2}}$$

Note the minus sign in the numerator!

Example 2 Consider $y = 1/\sin(x)$. The derivative may be found by writing $y = u/v$ where:

$$u = 1\,, \quad \Rightarrow \quad \frac{du}{dx} = 0 \qquad \text{and} \qquad v = \sin(x)\,, \quad \Rightarrow \quad \frac{dv}{dx} = \cos(x)$$

Inserting this into the **quotient rule** above yields:

$$\begin{aligned} \frac{dy}{dx} &= \frac{\sin(x) \times 0 - 1 \times \cos(x)}{\sin^2(x)} \\ &= -\frac{\cos(x)}{\sin^2(x)} \end{aligned}$$

Example 3 Consider $y = \tan(x) = \dfrac{\sin(x)}{\cos(x)}$. The derivative of the tangent may be found by writing $y = u/v$ where

$$u = \sin(x) \qquad\qquad v = \cos(x)$$

$$\Rightarrow \frac{du}{dx} = \cos(x) \qquad\qquad \Rightarrow \frac{dv}{dx} = -\sin(x)$$

Inserting this into the **quotient rule** yields:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\left(v\dfrac{du}{dx} - u\dfrac{dv}{dx}\right)}{v^2} \\
&= \frac{\cos(x) \times \cos(x) - \sin(x) \times (-\sin(x))}{\cos^2(x)} \\
&= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} \qquad \text{since} \quad \cos^2(x) + \sin^2(x) = 1
\end{aligned}$$

Exercise 4. Use the quotient rule to differentiate the functions below with respect to x (click on the **green** letters for the solutions).

(a) $y = u/v\,,$ if $u = \mathrm{e}^{ax}\,,$ and $v = \mathrm{e}^{bx}$

(b) $y = u/v\,,$ if $u = x+1\,,$ and $v = x-1$

Exercise 5. Use the quotient rule to differentiate the following with respect to x (click on the **green** letters for the solutions).

(a) $y = \sin(x)/(x+1)$

(b) $y = \sin(2x)/\cos(2x)$

(c) $y = (2x+1)/(x-2)$

(d) $y = \dfrac{\sqrt{x^3}}{3x+2}$

Quiz Select the derivative of $y = \cot(t)$ with respect to t.

(a) $\dfrac{-\sin(t)}{\cos(t)}$ (b) $-\dfrac{1}{\sin^2(t)}$ (c) $\dfrac{\cos^2(t)-\sin^2(t)}{\sin^2(t)}$ (d) $\dfrac{2\cos(t)\sin(t)}{\sin^2(t)}$

Hint: recall that the cotangent is given by $\cot(t) = \cos(t)/\sin(t)$

In the exercises and quiz below find the requested derivative by using the appropriate rule.

EXERCISE 6. **Differentiate** the following functions (click on the **green** letters for the solutions).

(a) $y = (z + 1)\sin(3z)$ with respect to z
(b) $y = 3(w^2 + 1)/(w + 1)$ with respect to w
(c) $W = \mathrm{e}^{2t}\ln(3t)$ with respect to t

Quiz The derivative, $\frac{dy}{dx}$, yields the rate of change of y with respect to x. Find the rate of change of $y = \dfrac{x}{x+1}$ with respect to x.

(a) $-\dfrac{1}{(x+1)^2}$ (b) 0 (c) $\dfrac{2x+1}{(x+1)^2}$ (d) $\dfrac{1}{(x+1)^2}$

4. Final Quiz

Begin Quiz Choose the solutions from the options given.

1. What is the derivative with respect to x of $y = x(\ln(x) - 1)$?

(a) $\ln(x) + \dfrac{1}{x}$ (b) $\ln(x)$ (c) 1 (d) $\dfrac{1}{x}$

2. Velocity is the derivative of position with respect to time. If the position, x, of a body is given by $x = 3t\mathrm{e}^{2t}$ (m) at time t (s), select its velocity from the answers below.

(a) $(6t + 3)\mathrm{e}^{2t}\,\mathrm{m\,s^{-1}}$ (b) $3 + 2\mathrm{e}^{2t}\,\mathrm{m\,s^{-1}}$
(c) $(3t + 2)\mathrm{e}^{2t}\,\mathrm{m\,s^{-1}}$ (d) $(6t^2 + 3)\mathrm{e}^{2t}\,\mathrm{m\,s^{-1}}$

3. Select below the rate of change of $y = (x^2 + 1)/(x^2 - 1)$ with respect to x.

(a) 1 (b) x
(c) $(4x^3)/(x^2 - 1)$ (d) $-4x/(x^2 - 1)^2$

End Quiz Score: Correct

Solutions to Exercises

Exercise 1(a) If $y = 4x^2 + 3x - 5$, then to calculate its derivative with respect to x, we need the sum rule and also the rule that

$$\frac{d}{dx}(ax^n) = nax^{n-1}$$

In the first term $a = 4$ and $n = 2$, in the second term $a = 3$ and $n = 1$ while the third term is a constant and has zero derivative. This yields

$$\begin{aligned} \frac{d}{dx}\left(4x^2 + 3x - 5\right) &= 2 \times 4x^{2-1} + 1 \times 3 \times x^{1-1} + 0 \\ &= 8x^1 + 3x^0 \\ &= 8x + 3 \end{aligned}$$

Here we used $x^0 = 1$.

Click on the **green** square to return

□

Exercise 1(b) To differentiate $y = 4\sin(3x)$ with respect to x we use the rule

$$\frac{d}{dx}(\sin(ax)) = a\cos(ax)$$

In this case with $a = 3$. We also take the derivative through the constant 4. This gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(4\sin(3x)) \\ &= 4\frac{d}{dx}(\sin(3x)) \\ &= 4 \times 3\cos(3x) \\ &= 12\cos(3x)\end{aligned}$$

Click on the **green** square to return

□

Exercise 1(c) To differentiate e^{-2x} with respect to x we need the rule

$$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$$

and here $a = -2$. This implies

$$\frac{d}{dx}\left(e^{-2x}\right) = -2e^{-2x}$$

Click on the **green** square to return □

Exercise 1(d) To differentiate $\ln\left(\frac{x}{2}\right)$ it is helpful to recall that $\log(A/B) = \log(A) - \log(B)$ (see the package on **Logarithms**) so

$$\ln\left(\frac{x}{2}\right) = \ln(x) - \ln(2)$$

The rule

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

together with the sum rule thus gives

$$\begin{aligned} \frac{d}{dx}\left(\ln(x) - \ln(2)\right) &= \frac{d}{dx}\left(\ln(x)\right) - \frac{d}{dx}\left(\ln(2)\right) \\ &= \frac{1}{x} - 0 \\ &= \frac{1}{x} \end{aligned}$$

since $\ln(2)$ is a constant and the derivative of a constant vanishes.

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Exercise 2(a) The function $y = x^m \times x^n = x^{m+n}$ (see the package on **Powers**). Thus the rule

$$\frac{d}{dx}(ax^n) = nax^{n-1}$$

tells us that

$$\frac{dy}{dx} = (m+n)x^{m+n-1}$$

This example also allows us to practise the **product rule**. From $y = x^m \times x^n$ the product rule yields

$$\begin{aligned} \frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= x^m \times nx^{n-1} + x^n \times mx^{m-1} \\ &= nx^{m+n-1} + mx^{m+n-1} \\ &= (m+n)x^{m+n-1} \end{aligned}$$

which is indeed the expected result.

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Exercise 2(b) To differentiate $y = 3x^4 \times e^{-2x}$ with respect to x we may use the results:

$$\frac{d}{dx}\left(3x^4\right) = 4 \times 3x^{4-1} \quad \text{and} \quad \frac{d}{dx}e^{-2x} = -2e^{-2x}$$

together with the **product rule**

$$\begin{aligned}
\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\
&= 3x^4 \times (-2e^{-2x}) + e^{-2x} \times 3 \times 4x^{4-1} \\
&= -6x^4e^{-2x} + 12x^3e^{-2x} \\
&= (-6x^4 + 12x^3)e^{-2x} \\
&= (-6x + 12)x^3e^{-2x} \\
&= 6(2 - x)x^3e^{-2x}
\end{aligned}$$

Click on the **green** square to return

$\square$

Exercise 2(c) To differentiate $y = x^3 \times \cos(x)$ with respect to x we may use the results:

$$\frac{d}{dx}x^3 = 3x^{3-1} \quad \text{and} \quad \frac{d}{dx}\cos(x) = -\sin(x)$$

together with the **product rule**

$$\begin{aligned} \frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= x^3 \times (-\sin(x)) + \cos(x) \times 3x^{3-1} \\ &= -x^3 \sin(x) + 3x^2 \cos(x) \\ &= x^2[3\cos(x) - x\sin(x)] \end{aligned}$$

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Exercise 2(d) To differentiate $y = \mathrm{e}^x \times \ln(x)$ with respect to x we may use the results:

$$\frac{d}{dx}\left(\mathrm{e}^x\right) = \mathrm{e}^x \quad \text{and} \quad \frac{d}{dx}\ln(x) = \frac{1}{x}$$

and the **product rule** to obtain

$$\begin{aligned} \frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= \mathrm{e}^x \times \frac{1}{x} + \ln(x) \times \mathrm{e}^x \\ &= \left[\frac{1}{x} + \ln(x)\right]\mathrm{e}^x \end{aligned}$$

Click on the **green** square to return □

Exercise 3(a) To differentiate $y = xe^{2x}$ with respect to x we rewrite y as: $y = uv$ where

$$\begin{array}{rrcl} & u = x & \text{and} & v = e^{2x} \\ \therefore & \dfrac{du}{dx} = 1 & \text{and} & \dfrac{dv}{dx} = 2e^{2x} \end{array}$$

Substituting this into the **product rule** yields

$$\begin{aligned} \frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= x \times 2e^{2x} + e^{2x} \times 1 \\ &= 2xe^{2x} + e^{2x} \\ &= (2x+1)e^{2x} \end{aligned}$$

Click on the **green** square to return $\square$

Exercise 3(b) To differentiate $y = \sin(x)\cos(2x)$ with respect to x we rewrite y as: $y = uv$ where

$$\begin{array}{rrcr} & u = \sin(x) & \text{and} & v = \cos(2x) \\ \therefore & \dfrac{du}{dx} = \cos(x) & \text{and} & \dfrac{dv}{dx} = -2\sin(2x) \end{array}$$

Substituting into the **product rule** gives

$$\begin{aligned} \frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= \sin(x) \times (-2\sin(2x)) + \cos(2x) \times \cos(x) \\ &= -2\sin(x)\sin(2x) + \cos(x)\cos(2x) \end{aligned}$$

Click on the **green** square to return

$\square$

Exercise 3(c) To differentiate $y = x\ln\left(4x^2\right)$ with respect to x we rewrite y as: $y = uv$ where

$$u = x \qquad \text{and} \qquad v = \ln\left(4x^2\right)$$

$$\therefore \quad \frac{du}{dx} = 1 \qquad \text{and} \qquad \frac{dv}{dx} = \frac{2}{x}$$

To obtain $\frac{dv}{dx}$ note that from the properties of logarithms: $\ln(4x^2) = \ln(4) + 2\ln(x)$ and recall that the derivative of $\ln(x)$ is $\frac{1}{x}$. Substituting this into the **product rule** gives

$$\begin{aligned} \frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= x \times \frac{2}{x} + \ln\left(4x^2\right) \times 1 \\ &= 2 + \ln\left(4x^2\right) \end{aligned}$$

Click on the **green** square to return $\square$

Exercise 3(d) To differentiate $y = \sqrt{x}\,\ln(x)$ with respect to x we rewrite y as: $y = uv$ where

$$u = \sqrt{x} = x^{\frac{1}{2}} \qquad \text{and} \qquad v = \ln(x)$$

$$\therefore \quad \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \qquad \text{and} \qquad \frac{dv}{dx} = \frac{1}{x}$$

Inserting this into the **product rule** implies

$$\begin{aligned}
\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\
&= x^{\frac{1}{2}} \times \frac{1}{x} + \ln(x) \times \frac{1}{2}x^{-\frac{1}{2}} \\
&= x^{\frac{1}{2}-1} + \frac{1}{2}x^{-\frac{1}{2}}\ln(x) \\
&= x^{-\frac{1}{2}}\left[1 + \frac{1}{2}\ln(x)\right]
\end{aligned}$$

Click on the **green** square to return □

Exercise 4(a) The function $y = \mathrm{e}^{ax}/\mathrm{e}^{bx} = \mathrm{e}^{(a-b)x}$ (see the package on **Powers**. Hence its derivative with respect to x is:

$$\frac{dy}{dx} = (a-b)\mathrm{e}^{(a-b)x}$$

This example can also be used to practise the **quotient rule**. From

$$u = \mathrm{e}^{ax} \quad \Rightarrow \quad \frac{du}{dx} = a\mathrm{e}^{ax} \qquad \text{and} \qquad v = \mathrm{e}^{bx} \quad \Rightarrow \quad \frac{dv}{dx} = b\mathrm{e}^{bx}$$

and the **quotient rule** one finds the expected result

$$\begin{aligned} \frac{dy}{dx} &= \frac{\left(v\dfrac{du}{dx} - u\dfrac{dv}{dx}\right)}{v^2} = \frac{\mathrm{e}^{bx} \times a\mathrm{e}^{ax} - \mathrm{e}^{ax} \times b\mathrm{e}^{bx}}{(\mathrm{e}^{bx})^2} \\ &= \frac{a\mathrm{e}^{ax+bx} - b\mathrm{e}^{ax+bx}}{\mathrm{e}^{2bx}} \\ &= \frac{(a-b)\mathrm{e}^{(a+b)x}}{\mathrm{e}^{2bx}} = (a-b)\mathrm{e}^{(a-b)x} \end{aligned}$$

Click on the **green** square to return $\square$

Exercise 4(b) To differentiate this function $y = u/v$ note that

$$u = x + 1 \quad \Rightarrow \quad \frac{du}{dx} = 1 \qquad \text{and} \qquad v = x - 1 \quad \Rightarrow \quad \frac{dv}{dx} = 1$$

and from the **quotient rule** one so obtains

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\left(v\dfrac{du}{dx} - u\dfrac{dv}{dx}\right)}{v^2} \\
&= \frac{(x-1) \times 1 - (x+1) \times 1}{(x-1)^2} \\
&= \frac{x - 1 - x - 1}{(x-1)^2} \\
&= \frac{-2}{(x-1)^2}
\end{aligned}$$

$\square$

Click on the **green** square to return

Exercise 5(a) To differentiate the function $y = \sin(x)/(x+1)$ write $y = u/v$ where

$$u = \sin(x) \quad \Rightarrow \quad \frac{du}{dx} = \cos(x) \qquad \text{and} \qquad v = x+1 \quad \Rightarrow \quad \frac{dv}{dx} = 1$$

and from the **quotient rule** one obtains

$$\begin{aligned} \frac{dy}{dx} &= \frac{\left(v\dfrac{du}{dx} - u\dfrac{dv}{dx}\right)}{v^2} \\ &= \frac{(x+1) \times \cos(x) - \sin(x) \times 1}{(x+1)^2} \\ &= \frac{(x+1)\cos(x) - \sin(x)}{(x+1)^2} \end{aligned}$$

Click on the **green** square to return □

Exercise 5(b) To differentiate $y = \sin(2x)/\cos(2x)$, let $y = u/v$ where

$$u = \sin(2x) \;\Rightarrow\; \frac{du}{dx} = 2\cos(2x) \quad \& \quad v = \cos(2x) \;\Rightarrow\; \frac{dv}{dx} = -2\sin(2x)$$

and from the **quotient rule** one obtains

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\left(v\dfrac{du}{dx} - u\dfrac{dv}{dx}\right)}{v^2} \\
&= \frac{\cos(2x) \times 2\cos(2x) - \sin(2x) \times (-2\sin(2x))}{\cos^2(2x)} \\
&= \frac{2\cos^2(2x) + 2\sin^2(2x)}{\cos^2(2x)} \\
&= \frac{2(\cos^2(2x) + \sin^2(2x))}{\cos^2(2x)} = \frac{2}{\cos^2(2x)}
\end{aligned}$$

since $\cos^2(\theta) + \sin^2(\theta) = 1$ for all angles θ.

Click on the **green** square to return □

Exercise 5(c) To differentiate the function $y = (2x+1)/(x-2)$ write $y = u/v$ where

$$u = 2x + 1 \quad \Rightarrow \quad \frac{du}{dx} = 2 \qquad \text{and} \qquad v = x - 2 \quad \Rightarrow \quad \frac{dv}{dx} = 1$$

and from the **quotient rule** one obtains

$$\begin{aligned} \frac{dy}{dx} &= \frac{\left(v\dfrac{du}{dx} - u\dfrac{dv}{dx}\right)}{v^2} \\ &= \frac{(x-2) \times 2 - (2x+1) \times 1}{(x-2)^2} \\ &= \frac{2x - 4 - 2x - 1}{(x-2)^2} \\ &= \frac{-5}{(x-2)^2} \end{aligned}$$

$\square$

Click on the **green** square to return

Exercise 5(d) To differentiate $y = \dfrac{\sqrt{x^3}}{3x+2}$, let $y = u/v$ where

$$u = \sqrt{x^3} = x^{\frac{3}{2}} \quad \Rightarrow \quad \frac{du}{dx} = \frac{3}{2}x^{\frac{1}{2}} \qquad \& \qquad v = 3x+2 \quad \Rightarrow \quad \frac{dv}{dx} = 3$$

and from the **quotient rule** one obtains

$$\begin{aligned} \frac{dy}{dx} &= \frac{(3x+2) \times \frac{3}{2}x^{\frac{1}{2}} - x^{\frac{3}{2}} \times 3}{(3x+2)^2} \\ &= \frac{\frac{9}{2}x^{\frac{3}{2}} + 3x^{\frac{1}{2}} - 3x^{\frac{3}{2}}}{(3x+2)^2} \\ &= \frac{(\frac{9}{2} - 3)x^{\frac{3}{2}} + 3x^{\frac{1}{2}}}{(3x+2)^2} \\ &= \frac{(\frac{3}{2})x^{\frac{3}{2}} + 3x^{\frac{1}{2}}}{(3x+2)^2} = \frac{3x^{\frac{1}{2}}(x+2)}{2(3x+2)^2} \end{aligned}$$

□

Click on the **green** square to return

Exercise 6(a) To differentiate $y = (z+1)\sin(3z)$ with respect to z we rewrite y as: $y = uv$ where

$$u = (z+1) \qquad \text{and} \qquad v = \sin(3z)$$

$$\therefore \quad \frac{du}{dz} = 1 \qquad \text{and} \qquad \frac{dv}{dz} = 3\cos(3z)$$

Substituting this into the **product rule** yields

$$\begin{aligned} \frac{dy}{dz} &= u\frac{dv}{dz} + v\frac{du}{dz} \\ &= (z+1) \times 3\cos(3z) + \sin(3z) \times 1 \\ &= 3(z+1)\cos(3z) + \sin(3z) \end{aligned}$$

Click on the **green** square to return

□

Exercise 6(b) To differentiate $y = 3(w^2+1)/(w+1)$, let $y = u/v$ where

$$u = 3(w^2+1) \quad \Rightarrow \quad \frac{du}{dw} = 6w \qquad \text{and} \qquad v = w+1 \quad \Rightarrow \quad \frac{dv}{dw} = 1$$

and from the **quotient rule** one obtains

$$\begin{aligned}
\frac{dy}{dw} &= \frac{\left(v\dfrac{du}{dw} - u\dfrac{dv}{dw}\right)}{v^2} \\
&= \frac{(w+1)\times 6w - 3(w^2+1)\times 1}{(w+1)^2} \\
&= \frac{6w^2+6w-3w^2-3}{(w+1)^2} \\
&= \frac{3w^2+6w-3}{(w+1)^2} = \frac{3(w^2+2w-1)}{(w+1)^2}
\end{aligned}$$

Click on the **green** square to return □

Exercise 6(c) To differentiate $W = \mathrm{e}^{2t}\ln(3t)$ with respect to t we rewrite W as: $W = uv$ where

$$u = \mathrm{e}^{2t} \qquad \text{and} \qquad v = \ln(3t) = \ln(3) + \ln(t)$$

$$\therefore \quad \frac{du}{dt} = 2\mathrm{e}^{2t} \qquad \text{and} \qquad \frac{dv}{dt} = \frac{1}{t}$$

Substituting this into the **product rule** yields

$$\begin{aligned} \frac{dW}{dt} &= u\frac{dv}{dt} + v\frac{du}{dt} \\ &= \mathrm{e}^{2t} \times \frac{1}{t} + \ln(3t) \times 2\mathrm{e}^{2t} \\ &= \left[\frac{1}{t} + 2\ln(3t)\right]\mathrm{e}^{2t} \end{aligned}$$

Click on the **green** square to return □

Solutions to Quizzes

Solution to Quiz:

To differentiate $y = \frac{1}{3}\mathrm{e}^{3t} - 3\cos\left(\frac{2t}{3}\right)$ with respect to t, we need the **sum rule** and the results

$$\frac{d}{dt}\left(\mathrm{e}^{at}\right) = a\mathrm{e}^{at}\,, \quad \& \quad \frac{d}{dt}\left(\cos(at)\right) = -a\sin(at)$$

This gives

$$\begin{aligned} \frac{dy}{dt} &= \frac{1}{3} \times 3\mathrm{e}^{3t} - 3 \times (-\frac{2}{3})\sin\left(\frac{2t}{3}\right) \\ &= \mathrm{e}^{3t} + 2\sin\left(\frac{2t}{3}\right) \end{aligned}$$

End Quiz

Solution to Quiz: The quotient rule may be used to differentiate $y = \cot(t)$ with respect to t. Writing $y = u/v$ with $u = \cos(t)$ and $v = \sin(t)$ this gives:

$$\begin{aligned}
\frac{dy}{dt} &= \frac{\left(v\dfrac{du}{dt} - u\dfrac{dv}{dt}\right)}{v^2} \\
&= \frac{\sin(t) \times (-\sin(t)) - \cos(t) \times (\cos(t))}{\sin^2(t)} \\
&= -\frac{\cos^2(t) + \sin^2(t)}{\sin^2(t)} \\
&= -\frac{1}{\sin^2(t)}
\end{aligned}$$

where we used $\cos^2(t) + \sin^2(t) = 1$ in the last step. End Quiz

Solution to Quiz: To differentiate $y = \dfrac{x}{x+1}$ with respect to x, we may use the quotient rule. For $y = u/v$ with $u = x$ and $v = x + 1$ this yields

$$\begin{aligned}\frac{dy}{dx} &= \frac{\left(v\dfrac{du}{dx} - u\dfrac{dv}{dx}\right)}{v^2} \\ &= \frac{(x+1)\times 1 - x \times (1)}{(x+1)^2} \\ &= \frac{1}{(x+1)^2}\end{aligned}$$

End Quiz